Chapter 1 : Introduction

1.1/ What is statistical mechanics ?

The 20th century has witnessed 3 scientific revolutions

- QM deals with the infinitely small (2022 Nobel prize on bell inequalities)
- GR deals with the infinitely large (2017 Nobel prize on gravitational waves)
- Statistical Mechanics deals with the infinitely complex (2021 Nobel prize for complex systems)

In a nutshell

A rather small number of atoms make up the entire world around us.

Q_1: How is it possible ?

Q_2: How do we account for this ?

"More is different" Phil Anderson, 1972.

A large number of interacting particles are able to

self-organize into states of matter with properties unmatched at the single-particle level.

Ex: emergence of the solid state

Q: Is accounting for this diversity hopeless?

Different is often the same: Many emerging phenomena look alike. Ex: Hexagonal packing in: Honey comb, pineapple, Giant causeway, cell tissue (eye of flies)

The recurrence of these patterns suggests the existence of underlying organizing principles. Deciphering them is the goal of statistical mechanics.

More can also be simpler: Poincaré taught us that the 3-body problem is not solvable & that classical mechanics with few interacting bodies is very hard.

$1\,m^3$ of air $\sim 10^{23}$ molecules $\Rightarrow$ horribly complicated!

But 3 numbers suffice to characterize it pretty well

$T = 22°C$, $P = 1$ bar, Humidity = 50%

3 numbers characterize pretty well 10^{23} degrees of freedom!

Goal of statistical mechanics: Identify which microscopic details are useless, eliminate them, and derive a self contained theory that accounts for the emerging properties of a system.

Comments: Nothing so far restricts the principle of statistical mechanics to physical systems ⇒ economy, sociology, epidemiology, computer science have all research fields at the interface with statistical physicists.

⚠ There are conditions for this to be wise (scale separation, universality, etc.)

This course: Focus on emerging properties of collections of particles

1.2) Equilibrium statistical mechanics: the ensemble approach

Most laws of physics are dynamical: Maxwell's equations, Einstein's equation, Schrödinger's equation, etc.

⇒ Very complicated PDEs that predict the state of a system given the perfect knowledge of its state at an earlier time.

Two limitations:

① Measuring the state of a system is very hard ($N \sim 10^{23}$)

② Solving these equations is very hard

Ensemble approach: Don't even try, it's useless.

Characterize the probability to find the system in a given configuration & more precisely its typical configurations and the fluctuations them.

Classical system $(\vec{q}, \vec{p}) \equiv (q_1, \dots, q_N, p_1, \dots, p_N)$

Characterize the probability density $\rho(\vec{q}, \vec{p})$ such that the probability to find the system in a volume

$$d\Gamma = d\vec{q}\,d\vec{p} = \prod_i dq_i\,dp_i$$

mean $\vec{q}, \vec{p}$ is $\rho(\vec{q},\vec{p})\,d\Gamma$

Equilibrium stat mech: right ansatz for $\rho(\vec{q},\vec{p})$ & use it to characterize the system.

1.2.1 | The microcanonical ensemble

Take an isolated classical system characterized by positions & momenta: $\Psi = \{q_1, \dots, q_N, p_1, \dots, p_N\}$ and a time independent Hamiltonian $H(q_1, \dots, q_N, p_1, \dots, p_N)$.

Dynamics Trajectories $q_i(t), p_i(t)$ are solutions of

$$\frac{d}{dt} q_i(t) \equiv \dot{q}_i(t) = \frac{\partial H}{\partial p_i} \equiv \partial_{p_i} H \,; \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$H(q_1(t), \dots, q_N(t), p_1(t), \dots, p_N(t))$ is a constant of motion:

Chain rule:

$$\frac{d}{dt} H(\vec{q}(t), \vec{p}(t)) = \underbrace{\frac{\partial H}{\partial t}}_{=0} + \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = \sum_i \partial_{q_i} H \times \partial_{p_i} H + \partial_{p_i} H \left[-\partial_{q_i} H\right] = 0$$

The dynamics of the system takes place along the energy surface.

Microcanonical hypothesis: For a classical complex system, the energy surface is visited uniformly & ergodically $\Rightarrow$ all

configurations with the same energy are visited with equal probability.